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# FLEXURAL VIBRATIONS OF A PIEZOELECTRIC BIMORPH WITH A CUT INTERNAL ELECTRODE

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*Stationary vibrations of a bimorph plate composed of two piezoelectric layers of equal thickness are studied. There is an infinitely thin cut electrode between the layers. A model of flexural vibrations of the bimorph that is based on the variational equation generalizing the Hamilton principle in electroelasticity is proposed. For the plane problem, a system of equations of motion is derived and the boundary conditions and the conjugate conditions at the interface of the regions of the cut electrode are formulated. For the TsTS-19 piezoceramics, resonance and antiresonance frequencies are calculated. The values obtained are compared with the calculation results obtained with the use of the Kirchhoff model and the finite-element method. It is shown that the use of a plate with a cut electrode allows one to increase the efficiency of vibration excitation compared to the case of a continuous internal electrode.*

The use of piezoelectric transducers in technical devices makes it necessary to develop models and methods of calculating the electric and mechanical fields in a piezoelectrically active medium.

Among studies dealing with the development of applied models of flexure of layered piezoelectric structures, the paper by Getman and Ustinov [1], in which the general regularities of deformation of inhomogeneous electroelastic plates are analyzed and a method of constructing a certain class of exact inhomogeneous solutions is proposed, is noteworthy.

The development of simplified models is important for practical calculation of electroelastic fields. Parton and Kudryavtsev [2] outlined the general scheme for studying the three-dimensional equations of electroelasticity. Using hypotheses on the distribution of electric and mechanical fields, Vatul'yan, Getman, and Lapitskaya [3] reduced the problem to the classical problem of bending.

As a rule, the models with continuous electrodes are employed to study vibrations of piezoelectric plates. In the present paper, a model of a piezoelectric bimorph plate with a cut internal electrode is proposed. Vatul'yan and Rynkova [4] treated a similar problem with the use of equations given in [3], where the Kirchhoff hypotheses for a piecewise homogeneous plate were used to obtain the classical vibration equations for the deflection of the middle surface of a bimorph plate, omitting the character of the electric-field distribution.

Vibrations of layered plates with cut electrodes are of interest in connection with the possibility of effective excitation of certain vibration modes. In the present paper, a model of flexural vibrations of a two-layer plate with a cut internal electrode is proposed. The model is constructed with the use of a variational equation for a piezoelectric medium that is the generalization of the Hamilton principle in the theory of elasticity.

We consider the plane problem of stationary flexural vibrations of a band plate which is infinite in the $x_2$ direction and consists of two equal-thickness piezoceramic layers polarized in the $x_3$ direction. The coordinate origin is at the middle surface of the plate. It is assumed that all the functions considered do not depend on the variable $x_2$.

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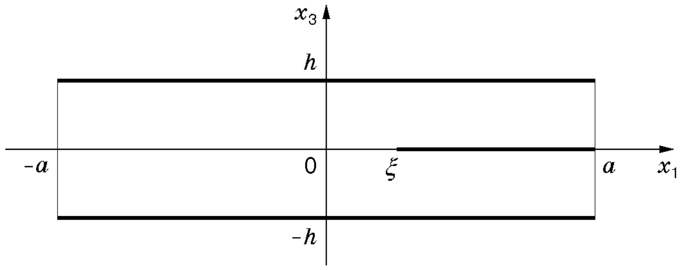


Fig. 1

Let there be electrodes on the faces of the plate $x_3 = \pm h$ and an infinitely thin cut electrode between the layers on the plane $x_3 = 0$. We denote the intersection of the plate and the middle surface by $\Omega = \Omega_1 \cup \Omega_2$, where $\Omega_1$ is the electrode-free region and $\Omega_2$ is the electrode-equipped region. Let $\Omega_1 = \{(x_1, x_2)\colon x_1 \in [-a, \xi],\ x_2 \in (-\infty, +\infty)\}$ and $\Omega_2 = \{(x_1, x_2)\colon x_1 \in [\xi, a],\ x_2 \in (-\infty, +\infty)\}$, where $\xi \in (-a, a)$ is the coordinate of the point separating these regions (Fig. 1). Vibrations are excited by the difference between the potentials on the faces and the cut internal electrode: $\varphi\Big|_{x_3=0} = V_0\,\mathrm{e}^{i\omega t}, x_1 \in \Omega_2$ and $\varphi\Big|_{x_3=\pm h} = 0$, where $\varphi$ is the electric potential, $\omega$ is the vibration frequency, and $V_0 = \text{const}$ is the specified amplitude.

Vibrations of the plate are governed by the equations [2]

$$\sigma_{ij,j} = -\rho\omega^2 u_i, \qquad D_{i,i} = 0,$$

in which $\sigma_{ij}$ are the stress-tensor components, $D_i$ are the components of the electric-induction vector, $i$, $j = 1$, 3, and $\rho$ is the density of piezoceramics. It is assumed that the lateral surface of the plate is stress-free: $\sigma_{11} = 0$ and $\sigma_{13} = 0$ for $x_1 = \pm a$. The faces $x_3 = \pm h$ are free from loads: $\sigma_{13} = \sigma_{33} = 0$. Let air be the ambient medium and, hence, $D_1 = 0$ for $x_1 = \pm a$.

The governing relations for an electroelastic medium polarized in the $x_3$ direction have the form [2]

$$\begin{gathered}\sigma_{11} = c_{11}^E \varepsilon_{11} + c_{13}^E \varepsilon_{33} + e_{31}\varphi_{,3}, \quad \sigma_{33} = c_{13}^E \varepsilon_{11} + c_{33}^E \varepsilon_{33} + e_{33}\varphi_{,3}, \quad \sigma_{13} = 2c_{44}^E \varepsilon_{13} + e_{31}\varphi_{,1}, \\ D_1 = 2e_{15}\varepsilon_{13} - \epsilon_{11}^S \varphi_{,1}, \qquad D_3 = e_{31}\varepsilon_{11} + e_{33}\varepsilon_{33} + \epsilon_{33}^S \varphi_{,3}.\end{gathered} \tag{1}$$

Here $\varepsilon_{ij}$ are the strain-tensor components, $c_{ij}^E$ are the moduli of elasticity measured in the presence of a constant electric field, $e_{31}$, $e_{33}$, and $e_{15}$ are the piezoelectric constants, and $\epsilon_{11}^S$ and $\epsilon_{33}^S$ are the permittivities for constant strains.

We now make some simplifications. We use the Kirchhoff hypotheses on the displacement distribution over the thickness: $u_1(x_1, x_3) = -x_3 w_{,1}$ and $u_3(x_1, x_3) = w(x_1)$, where $w(x_1)$ is the deflection of the middle surface of the plate. In accordance with the Kirchhoff hypotheses, we assume that the normal stress $\sigma_{33} = 0$ everywhere in the region occupied by the plate. Using the constitutive relation for $\sigma_{33}$, we eliminate the strain $\varepsilon_{33}$ from the equation of state of the piezoelectric medium (1):

$$\sigma_{11} = c_{11}^* u_{1,1} + e_{31}^* \varphi_{,3}, \qquad D_3 = e_{31}^* u_{1,1} - \epsilon_{33}^* \varphi_{,3}. \tag{2}$$

Here $c_{11}^* = c_{11}^E - (c_{13}^E)^2/c_{33}^E$, $e_{31}^* = e_{31} - c_{13}^E e_{33}/c_{33}^E$, and $\epsilon_{33}^* = \epsilon_{33}^S + e_{33}^2/c_{33}^E$. The expressions for $\sigma_{13}$ and $D_1$ remain the same.

In contrast to [3, 4], where the problem is reduced to an equation of one function, we introduce two functions. Let the function $V(x_1)$ be a value of the electric potential $\varphi(x_1, x_3)$ on the middle surface $x_3 = 0$. In the electrode-free region $\Omega_1$, this function is unknown, whereas in the region $\Omega_2$, we have $V(x_1) = V_0$. Obviously, the electric-potential function is continuous. However, in the presence of the internal electrode, the derivative $\varphi_{,3}$ has a discontinuity at $x_3 = 0$. Therefore, at the central electrode $x_3 = 0$, the normal component of the electric-induction vector $D_3$ has a jump denoted by $q(x_1) = D_3(x_1, +0) - D_3(x_1, -0)$. We note that in the region $\Omega_1$, the function $D_3$ is continuous, and, hence, $q(x_1) \equiv 0$ therein.

We assume that the electric potential is characterized by the following distribution over the $x_3$ coordinate:

$$\varphi(x_1,x_3) = \Big(1 - \frac{x_3^2}{h^2}\Big)V(x_1) + \frac{h}{2\epsilon_{33}^*}\Big(\frac{x_3^2}{h^2} - \frac{|x_3|}{h}\Big)q(x_1). \tag{3}$$

The potential taken in the form (3) satisfies automatically the boundary conditions at the faces $x_3 = \pm h$ and the internal electrode and the discontinuity condition for $\varphi_{,3}$.

We use the variational equation that generalizes the Hamilton principle in the theory of electroelasticity. In the case where the vibrations are stationary, and the mass forces, the surface loads, and the surface charges are absent, for plane strain, this equation has the form [2]

$$\int_{-a}^{a}\int_{-h}^{h} \delta H\, dx_3\, dx_1 - \rho\omega^2 \int_{-a}^{a}\int_{-h}^{h} u_i \delta u_i\, dx_3\, dx_1 = 0. \tag{4}$$

The electric enthalpy $H = U - E_i D_i$, where $U$ is the internal energy, is a function of strains $\varepsilon_{ij}$ and electric field $E_i$. The enthalpy variation is given by $\delta H = \sigma_{ij}\delta\varepsilon_{ij} - D_i \delta E_i$. With allowance for the above hypotheses, we obtain

$$\delta H = \sigma_{11}\delta\varepsilon_{11} - D_1\delta E_1 - D_3 \delta E_3. \tag{5}$$

We now calculate the stress-tensor and the electric-induction vector components in each region by formulas (1) and (2) and substitute the resulting expressions into (5). Let $\delta H^{\mathrm{I}}$ and $\delta H^{\mathrm{II}}$ be the enthalpy variations in the regions $\Omega_1$ and $\Omega_2$, respectively. We calculate the variation $\delta H^{\mathrm{I}}$ and $\delta H^{\mathrm{II}}$ in the segments $[-a,\xi]$ and $[\xi,a]$, respectively, and integrate the resulting expressions over the thickness. Using the variational equation (4), we obtain equations which involve the independent variations $\delta w^{\mathrm{I}}$ and $\delta V$ in the region $\Omega_1$ and $\delta w^{\mathrm{II}}$ and $\delta q$ in the region $\Omega_2$. Equating the coefficients of the independent variations to zero, we obtain the following systems of differential equations:

$$a_{11}\frac{d^4 w^{\mathrm{I}}}{dx_1^4} + a_{12}\frac{d^2 V}{dx_1^2} + \frac{2}{3}h^3\rho\omega^2\frac{d^2 w^{\mathrm{I}}}{dx_1^2} - 2h\rho\omega^2 w^{\mathrm{I}} = 0,$$

$$a_{21}\frac{d^2 w^{\mathrm{I}}}{dx_1^2} + a_{22}\frac{d^2 V}{dx_1^2} - a_{23}V = 0, \qquad x_1 \in \Omega_1,$$

$$a_{11}\frac{d^4 w^{\mathrm{II}}}{dx_1^4} - a_{13}\frac{d^2 q}{dx_1^2} + \frac{2}{3}h^3\rho\omega^2\frac{d^2 w^{\mathrm{II}}}{dx_1^2} - 2h\rho\omega^2 w^{\mathrm{II}} = 0,$$

$$a_{31}\frac{d^2 w^{\mathrm{II}}}{dx_1^2} - a_{32}\frac{d^2 q}{dx_1^2} + a_{33}q - \frac{1}{3}V_0 = 0, \qquad x_1 \in \Omega_2.$$

The coefficients in these equations depend on the physical constants of the material and the geometrical dimensions of the plate:

$$a_{11} = (2/3)h^3 c_{11}^*, \qquad a_{12} = a_{21} = (4/3)h e_{31}^*, \qquad a_{22} = (16/15)h\epsilon_{11}^S,$$

$$a_{23} = \frac{8}{3h}\epsilon_{33}^*, \qquad a_{13} = a_{31} = \frac{h^2}{6}\frac{e_{31}^*}{\epsilon_{33}^*}, \qquad a_{32} = \frac{h^3}{60}\frac{\epsilon_{11}^S}{(\epsilon_{33}^*)^2}, \qquad a_{33} = \frac{1}{6}\frac{h}{\epsilon_{33}^*}.$$

Equating the coefficients of the independent variations in the integrated terms, we obtain the boundary conditions

$$a_{11}\frac{d^3 w^{\mathrm{I}}}{dx_1^3} + a_{12}\frac{dV}{dx_1} + \frac{2}{3}h^3\rho\omega^2\frac{dw^{\mathrm{I}}}{dx_1} = 0, \quad a_{11}\frac{d^2 w^{\mathrm{I}}}{dx_1^2} + a_{12}V = 0, \quad \frac{dV}{dx_1} = 0 \quad \text{for } x_1 = -a,$$

$$a_{11}\frac{d^3 w^{\mathrm{II}}}{dx_1^3} - a_{13}\frac{dq}{dx_1} + \frac{2}{3}h^3\rho\omega^2\frac{dw^{\mathrm{II}}}{dx_1} = 0, \ a_{11}\frac{d^2 w^{\mathrm{II}}}{dx_1^2} + a_{12}V_0 - a_{13}q = 0, \ \frac{dq}{dx_1} = 0 \ \text{for } x_1 = -a.$$

As the conjugation conditions at the boundary of the regions $x_1 = \xi$, we require that the deflection and the slope be continuous $w^{\mathrm{I}} = w^{\mathrm{II}}$ and $dw^{\mathrm{I}}/dx_1 = dw^{\mathrm{II}}/dx_1$. Equating the coefficients of the variations $\delta w^{\mathrm{I}}$ and

TABLE 1

| $\xi$ | $l$ | Variational principle | | Kirchhoff theory | | FEM | |
|---|---|---|---|---|---|---|---|
| | | $\omega_r^*$ | $\omega_a^*$ | $\omega_r^*$ | $\omega_a^*$ | $\omega_r^*$ | $\omega_a^*$ |
| | | First mode | | | | | |
| −0.5 | 0.75 | 1.0338 | 1.1152 | 0.9729 | 0.9834 | 1.0175 | 1.0422 |
| 0.1 | 0.45 | 1.0743 | 1.1108 | 0.9902 | 0.9939 | 1.0595 | 1.0672 |
| 0.8 | 0.10 | 1.1152 | 1.1155 | 1.0007 | 1.0008 | 1.0988 | 1.0989 |
| | | Second mode | | | | | |
| −0.5 | 0.75 | 2.8005 | 2.8056 | 2.6717 | 2.6718 | 2.6561 | 2.6592 |
| 0.1 | 0.45 | 2.8879 | 2.9940 | 2.7214 | 2.7363 | 2.7423 | 2.7649 |
| 0.8 | 0.10 | 2.9931 | 2.9972 | 2.7704 | 2.7705 | 2.8498 | 2.8508 |

**Note.** $\omega_r^* = \omega_r \cdot 10^{-5}$ Hz and $\omega_a^* = \omega_a \cdot 10^{-5}$ Hz.

$\delta w^{\rm II}$, $d\delta w^{\rm I}/dx_1$ and $d\delta w^{\rm II}/dx_1$, and $\delta q$ in the integrated terms in the variational equations, we obtain three matching conditions:

$$a_{11}\Big(\frac{d^3w^{\rm I}}{dx_1^3} - \frac{d^3w^{\rm II}}{dx_1^3}\Big) + \frac{2}{3}\,h^3\rho\omega^2\Big(\frac{dw^{\rm I}}{dx_1} - \frac{dw^{\rm II}}{dx_1}\Big) + a_{12}\,\frac{dV}{dx_1} + a_{13}\,\frac{dq}{dx_1} = 0,$$

$$a_{11}\Big(\frac{d^2w^{\rm I}}{dx_1^2} - \frac{d^2w^{\rm II}}{dx_1^2}\Big) + a_{12}(V - V_0) - a_{13}q = 0, \qquad \frac{dq}{dx_1} = 0.$$

Finally, we require that the potential be continuous in passing through the boundary: $V = V_0$ for $x_1 = \xi$.

Thus, we have obtained two systems of differential equations for the following four desired functions: the deflection $w^{\rm I}$ and the potential $V$ in the region $\Omega_1$ and the deflection $w^{\rm II}$ and the jump in the normal component of the electric-induction vector on the middle surface $q$ in the region $\Omega_2$. Each system contains a fourth-order equation for the deflection function.

The above-constructed model enables one to calculate the resonance frequencies $\omega_r$. The antiresonance frequencies $\omega_a$ are determined under the condition that the current passing through the electrode vanishes: $I = -i\omega \int\limits_S D_3\,ds = 0$. As the surface $S$, we use the face electrode $x_3 = h$. In this case, the condition for determining the antiresonance frequencies takes the form

$$\int\limits_{-a}^{\xi} \Big(he_{31}^*\,\frac{d^2w^{\rm I}}{dx_1} - \frac{2}{h}\,\epsilon_{33}^* V\Big)dx_1 + \int\limits_{\xi}^{a} \Big(he_{31}^*\,\frac{d^2w^{\rm II}}{dx_1} - \frac{2}{h}\,\epsilon_{33}^* V_0 + \frac{1}{2}\,q\Big)dx_1 = 0.$$

We solved the formulated problem for a TsTS-19 plate with the thickness-to-width ratio $h/a = 0.1$ [5]. The frequencies $\omega_r$ and $\omega_a$ were calculated for various values of $\xi$. The efficiency of vibrations was estimated on the basis of the magnitude of the dynamic electromechanical-coupling coefficient [2] $k_d^2 = (\omega_a^2 - \omega_r^2)/\omega_a^2$.

The vibration frequencies calculated by the model proposed are compared with the theoretical results [4] based on the classical Kirchhoff hypotheses and finite-element results for an electroelastic rectangle. Table 1 lists some resonance $\omega_r$ and antiresonance $\omega_a$ frequencies for the first two vibration modes [$l = (a-\xi)/(2a)$ is the relative length of the electrode-equipped surface $\Omega_2$]. The frequencies increase monotonically as the length of the electrode $l$ decreases, the antiresonance frequencies being smaller than the resonance frequencies. These results are in agreement with the theoretical statements given in [6] and the numerical results obtained in [4].

The first frequencies predicted by the above model and by the model of [4] differ from those calculated by the finite-element method by 1–2 and 4–9%, respectively. The errors in determining the second frequencies

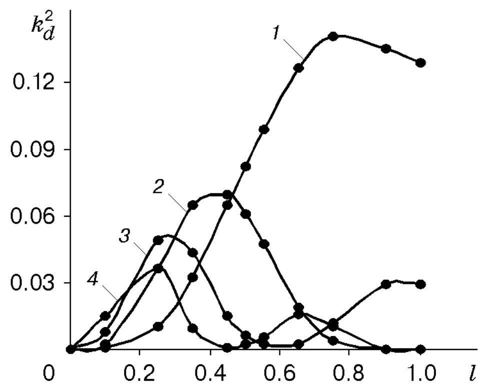


Fig. 2

relative to the finite-element results are 4–5% and 1–3% according to the above-considered theory and the theory of [4], respectively.

Figure 2 shows the dynamic coefficient of electromechanical coupling $k_d^2$ versus the relative length of the electrode $l$ for the first four vibration modes (curves 1–4). The points refer to the model proposed. The value $l = 1$ corresponds to the case of a continuous electrode. It follows from the results that, for each vibration mode, the internal electrode can be cut in such a manner that the efficiency of vibration excitation is higher than that for a continuous electrode. For the first mode, the coefficient $k_d^2$ can be increased by approximately 9% for a relative length of the internal electrode equal to 0.70–0.85. In exciting higher modes, one can increase the efficiency by decreasing the length of the electrode and by choosing it in the range from 0.3 to 0.5 for the second mode and from 0.2 to 0.4 for the third and fourth vibration modes.

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